Section 16.1
Vector Fields

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1 Vector Field Basics

Vector Fields

Goal: Describe physical phenomena such as current, wind direction, electric and magnetic fields that vary over space.

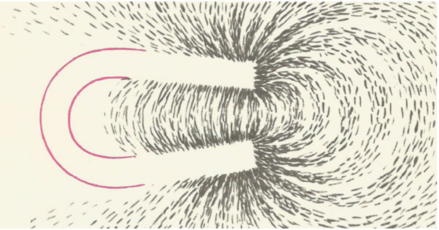
Definition: Vector Fields

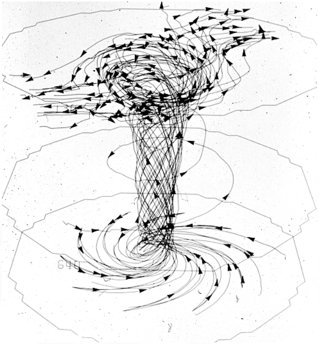
A **vector field** in $\mathbb{R}^n$ is a function $\vec{F} : \mathbb{R}^n \to \mathbb{R}^n$.

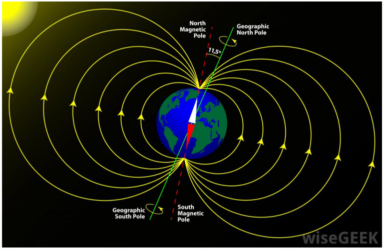
That is, $\vec{F}$ assigns to each point $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ a vector

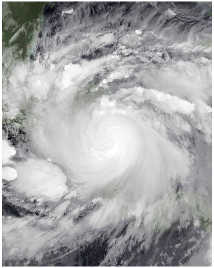
$$\vec{F}(x_1, x_2, \ldots, x_n) \;=\; \langle F_1(x_1, \ldots, x_n),\ \ldots,\ F_n(x_1, \ldots, x_n)\rangle$$

where $F_1, \ldots, F_n$ are scalar functions (the **component functions** of $\vec{F}$).









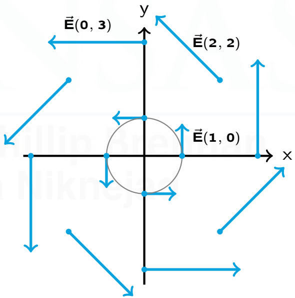
Hurricane Julia
10/09/2022
en.wikipedia.org

Picturing Vector Fields

Example 1, Part a: Sketch the vector field $\vec{E}$ in $\mathbb{R}^2$ defined by

$$\vec{E}(x, y) = \langle -y, x \rangle$$

(x,y)	$\vec{E}(x, y)$	(x,y)	$\vec{E}(x, y)$
$(1, 0)$	$\langle 0, 1 \rangle$	$(0, -3)$	$\langle 3, 0 \rangle$
$(3, 0)$	$\langle 0, 3 \rangle$	$(0, -1)$	$\langle 1, 0 \rangle$
$(2, 2)$	$\langle -2, 2 \rangle$	$(-1, 0)$	$\langle 0, -1 \rangle$
$(0, 3)$	$\langle -3, 0 \rangle$	$(-3, 0)$	$\langle 0, -3 \rangle$
$(0, 1)$	$\langle -1, 0 \rangle$	$(-2, 2)$	$\langle -2, -2 \rangle$
$(2, -2)$	$\langle 2, 2 \rangle$	$(-2, -2)$	$\langle 2, -2 \rangle$

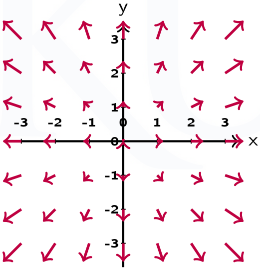


(As you can see, drawing vector fields by hand is a major hassle.)

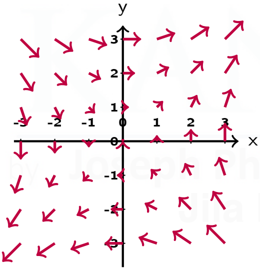
Picturing Vector Fields

Example 1, Part b: Here are three more vector fields in $\mathbb{R}^2$.

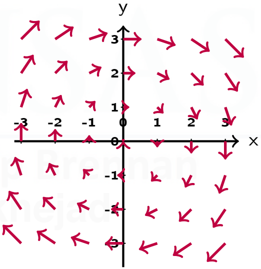
$\vec{F}(x,y) = x\vec{i} + y\vec{j}$.



$\vec{G}(x,y) = y\vec{i} + x\vec{j}$



$\vec{H}(x,y) = y\vec{i} - x\vec{j}$

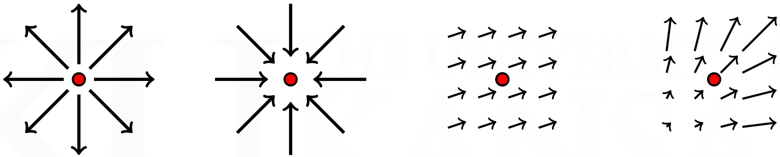


Note: $\vec{F}$ is a **radial vector field**: $\vec{F}(P)$ depends only on the distance from P to the origin O, and is parallel to $\overrightarrow{OP}$.

2 Divergence, Curl, and the Del Operator

Divergence of a Vector Field

The **divergence** of a vector field $\vec{F}$ at a point P measures how much $\vec{F}$ disperses "stuff" near P.



Positive divergence (disperses stuff) | Negative divergence (attracts stuff) | Zero divergence ("incompressible") | Positive divergence

The **divergence** of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is defined as

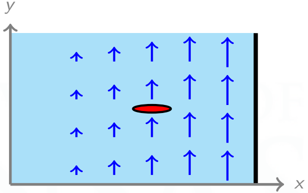
$$\operatorname{div}(\vec{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Notice that $\operatorname{div}(\vec{F})$ is a scalar-valued function.

Curl of a Vector Field

The **curl** of a vector field $\vec{F}$ measures how $\vec{F}$ causes objects to rotate.

Thought experiment: The current in a river is stronger near the banks than in the middle. A boat is anchored near the right bank. What happens to the boat? **It rotates counterclockwise.**



Let $\vec{F}$ be the vector field describing the current. Rotation occurs because the $\vec{j}$ component of $\vec{F}$ gets bigger to the right. That is,

$$\frac{\partial F_2}{\partial x}(\vec{a}) > 0.$$

- If $\frac{\partial F_2}{\partial x}(\vec{a}) < 0$ then $\vec{F}$ tends to rotate objects clockwise.
- The axis of rotation is parallel to the z-axis.
- The value of $\frac{\partial F_1}{\partial y}(\vec{a})$ also causes rotation (counterclockwise if negative, clockwise if positive).

Link

Curl of a Vector Field

- The tendency of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ to rotate objects counterclockwise around the z-axis is measured by the scalar quantity

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}.$$

- Correspondingly, rotation about the x- and y-axes are measured by the scalars

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \qquad \text{and} \qquad \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}.$$

- Making these scalar functions into the components of a **vector** lets us measure the rotational effect of $\vec{F}$ at all points.

Curl of a Vector Field

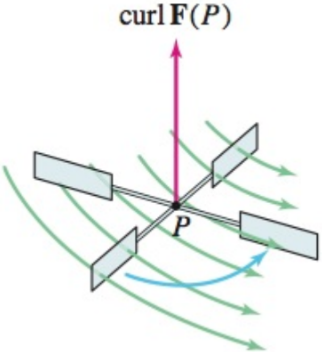
The **curl** of a vector field $\vec{F} = \langle F_1, F_2, F_3 \rangle$ is defined as

$$\operatorname{curl}(\vec{F}) = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x},\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

Notice that $\operatorname{curl}(\vec{F})$ is a **vector-valued** function (that is, it is a vector field).

The **direction** of $\operatorname{curl}(\vec{F})(P)$ is the axis of rotation, as determined by the right-hand rule, and the **magnitude** of $\operatorname{curl}(\vec{F})(P)$ is the speed of rotation.

If $\operatorname{curl}(\vec{F}) = \vec{0}$ then $\vec{F}$ is called **irrotational.**



The Del Operator

The **del** or **nabla** operator[1] ∇ is defined by $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

Applying ∇ to a scalar function f gives its gradient:

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The curl and divergence of a vector field can also be written in terms of ∇:

$$\text{div}(\vec{\mathsf{F}}) = \nabla \cdot \vec{\mathsf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle F_1, F_2, F_3 \rangle = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\text{curl}(\vec{\mathsf{F}}) = \nabla \times \vec{\mathsf{F}} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle = \begin{vmatrix} \vec{\mathsf{i}} & \vec{\mathsf{j}} & \vec{\mathsf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

[1] An **operator** is like a function on functions — it transforms one function into another.

Calculating Divergence and Curl

Example 2: Calculate the divergence and curl of

$$\vec{F}(x, y, z) = \left\langle xz,\ xyz,\ -y^2 \right\rangle.$$

Solution:

$$\operatorname{div}(\vec{F})(x, y, z) = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz$$

$$\operatorname{curl}(\vec{F})(x, y, z) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix} = \langle -2y - xy, x, yz \rangle$$

Calculating Divergence and Curl

Example 3: Calculate $\nabla \cdot \vec{F}$ and $\nabla \times \vec{F}$ for the 2-dimensional vector field

$$\vec{F}(x,y) = \left\langle y^2,\ x^2 \right\rangle .$$

<u>Solution:</u> This field turns out to be incompressible, because

$$(\nabla \cdot \vec{F})(x,y) = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(x^2) = 0.$$

Cross products are defined only in $\mathbb{R}^3$. To calculate curl we must write

$$\vec{F}(x,y) = \left\langle y^2,\ x^2,\ 0 \right\rangle$$

so that

$$(\nabla \times \vec{F})(x,y) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & 0 \end{vmatrix} = \langle 0,\ 0,\ 2x - 2y \rangle .$$

Fact: The curl of a 2-dimensional vector field is always parallel to $\vec{k}$.

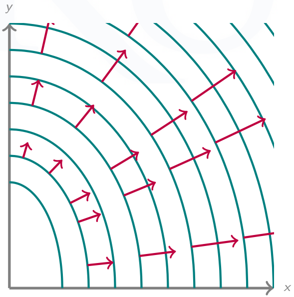
3 Conservative Vector Fields and their Potential Functions

Conservative Vector Fields

Let $f(x, y, z)$ be a scalar-valued function. Its gradient is a vector field:

$$\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x},\ \frac{\partial f}{\partial y},\ \frac{\partial f}{\partial z} \right\rangle$$

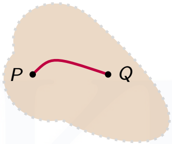
- The function f is called a **(scalar) potential function** for $\vec{F}$.
- A vector field is called **conservative** if it has a potential function,



Conservative fields occur naturally in physics, as force fields in which energy is conserved.

If $\vec{F} = \nabla f$ is a conservative vector field, then at all points P the vector $\vec{F}(P)$ is orthogonal to the level curve of the potential function f.

Facts about Potentials



A domain $\mathcal{R}$ is called **connected** if any two points P, Q in $\mathcal{R}$ can be connected by a path that lies in $\mathcal{R}$.

Theorem

If $\vec{F}$ is conservative on an open connected domain $\mathcal{R}$, then any two potential functions of $\vec{F}$ differ by a constant.

- This fact makes sense if you think of ∇ as differentiation and a potential function as an antiderivative.
- It is the higher-dimensional analogue of the statement that any two antiderivatives of a function $f : [a, b] \to \mathbb{R}$ differ by a constant.

Conservative Vector Fields Have Zero Curl

Theorem

If $\vec{F}$ is a conservative vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$, then $\text{curl}(\vec{F}) = \vec{0}$.

That is, **all conservative vector fields are irrotational.**

Proof: Let f be a potential function for $\vec{F}$, that is, $\nabla f = \vec{F}$. Then,

$$\text{curl}(\vec{F}) = \text{curl}(\nabla f) = \nabla \times \nabla f = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_x & f_y & f_z \end{vmatrix}$$

$$= \langle f_{zy} - f_{yz},\ f_{xz} - f_{zx},\ f_{yx} - f_{xy} \rangle$$

$$= \vec{0}, \qquad \text{by Clairaut's Theorem.}$$

Food For Thought: Are all irrotational fields necessarily conservative?

Finding Scalar Potentials

The process for finding scalar potential functions is essentially antidifferentiation, but with a twist.

For $\vec{F}(x, y) = \langle F_1(x, y),\ F_2(x, y)\rangle$:

1. Find the indefinite integrals $\int F_1(x, y)\,dx$ and $\int F_2(x, y)\,dy$.
   - **The constants of integration are $c_1(y)$ and $c_2(x)$ respectively** (instead of the usual "$+C$"), because if $\frac{\partial}{\partial x}(f(x, y)) = F_1$ then $\frac{\partial}{\partial x}(f(x, y) + c_1(y)) = F_1$ as well.
2. "Match up the pieces" to determine $f(x, y)$.

For $\vec{F}(x, y, z) = \langle F_1(x, y, z),\ F_2(x, y, z),\ F_3(x, y, z)\rangle$:

1. Find the indefinite integrals $\int F_1\,dx$, $\int F_2\,dy$, and $\int F_3\,dz$. Constants of integration: $c_1(y, z)$, $c_2(x, z)$, $c_3(x, y)$.
2. "Match up the pieces" to determine $f(x, y, z)$.

Finding Scalar Potentials

Example 4: Find a scalar potential function for the vector field

$$\vec{F}(x, y) = \left\langle 3 + 2xy,\ x^2 - 3y^2 \right\rangle .$$

<u>Solution:</u>

$$f(x, y) = \int 3 + 2xy \, dx = 3x + x^2 y + c_1(y)$$

$$f(x, y) = \int x^2 - 3y^2 \, dy = x^2 y - y^3 + c_2(x)$$

Match up the pieces:

$$f(x, y) = x^2 y + 3x - y^3 + C.$$

Finding Scalar Potentials (3-dimentional Example)

Example 5: Find a scalar potential function for the vector field

$$\vec{F}(x, y, z) = \left\langle y^2 + e^z,\ 2xy + \sec^2(y),\ xe^z \right\rangle.$$

Solution: Antidifferentiate each of the component functions:

$$\int y^2 + e^z\, dx = xy^2 + xe^z + c_1(y, z)$$

$$\int 2xy + \sec^2(y)\, dy = xy^2 + \underbrace{\tan(y)}_{c_1(y,z),\ c_3(x,y)} + c_2(x, z)$$

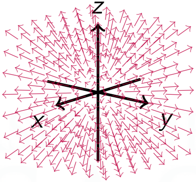
$$\int xe^z\, dz = xe^z + c_3(x, y)$$

Match up the pieces to get the answer:

$$f(x, y, z) = xy^2 + xe^z + \tan(y) + C.$$

Another Potential 3-Dimensional Example (Optional)

Example 6: Show that $r = \sqrt{x^2+y^2+z^2}$ is a potential function for the unit radial vector field



$$\vec{e}_r = \left\langle \frac{x}{r},\ \frac{y}{r},\ \frac{z}{r} \right\rangle .$$

<u>Solution:</u> $\dfrac{\partial r}{\partial x} = \dfrac{x}{\sqrt{x^2+y^2+z^2}} = \dfrac{x}{r} \qquad \dfrac{\partial r}{\partial y} = \dfrac{y}{r} \qquad \dfrac{\partial r}{\partial z} = \dfrac{z}{r}$

Radial, inverse-squared vector fields are conservative since

$$\nabla\left(\frac{-1}{r}\right) = \frac{\vec{e}_r}{r^2} \qquad\qquad \vec{F}_{gravity} = \left(\frac{-GmM}{r^2}\right)\vec{e}_r$$

Gravitational force exerted by a point mass m on a point mass M is described by a radial, inverse-squared vector field. $\dfrac{GmM}{r}$ is a scalar potential for $\vec{F}_{gravity}$.